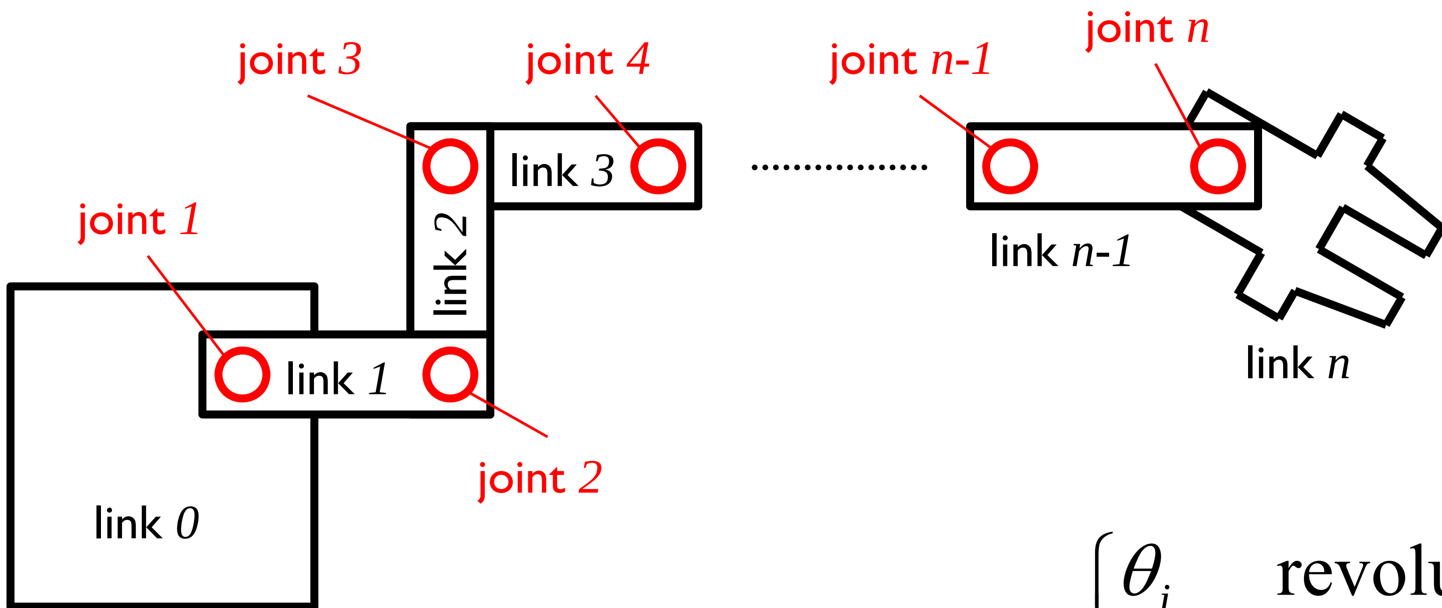
Forward Kinematics

Links and Joints

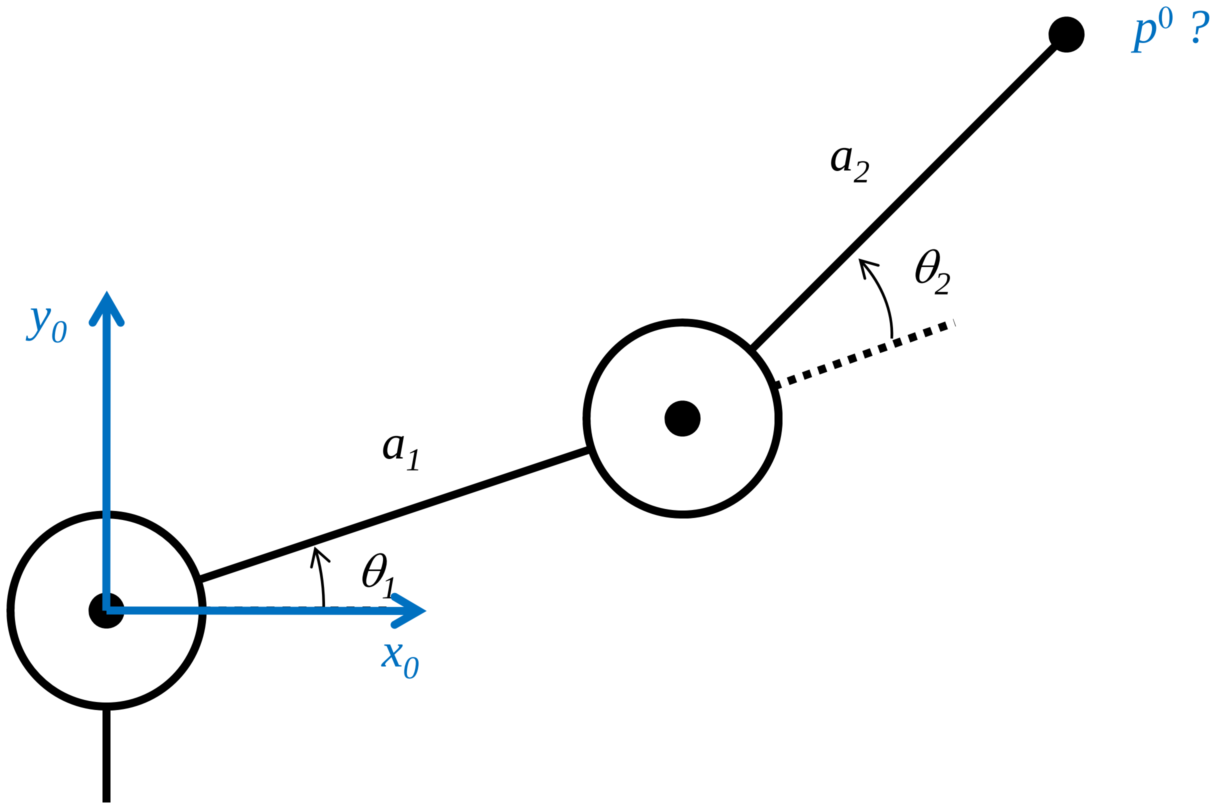


$$q_i = \begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$$

- n joints, $n + 1$ links
- link 0 is fixed (the base)
- joint i connects link $i - 1$ to link i
  - link i moves when joint i is actuated

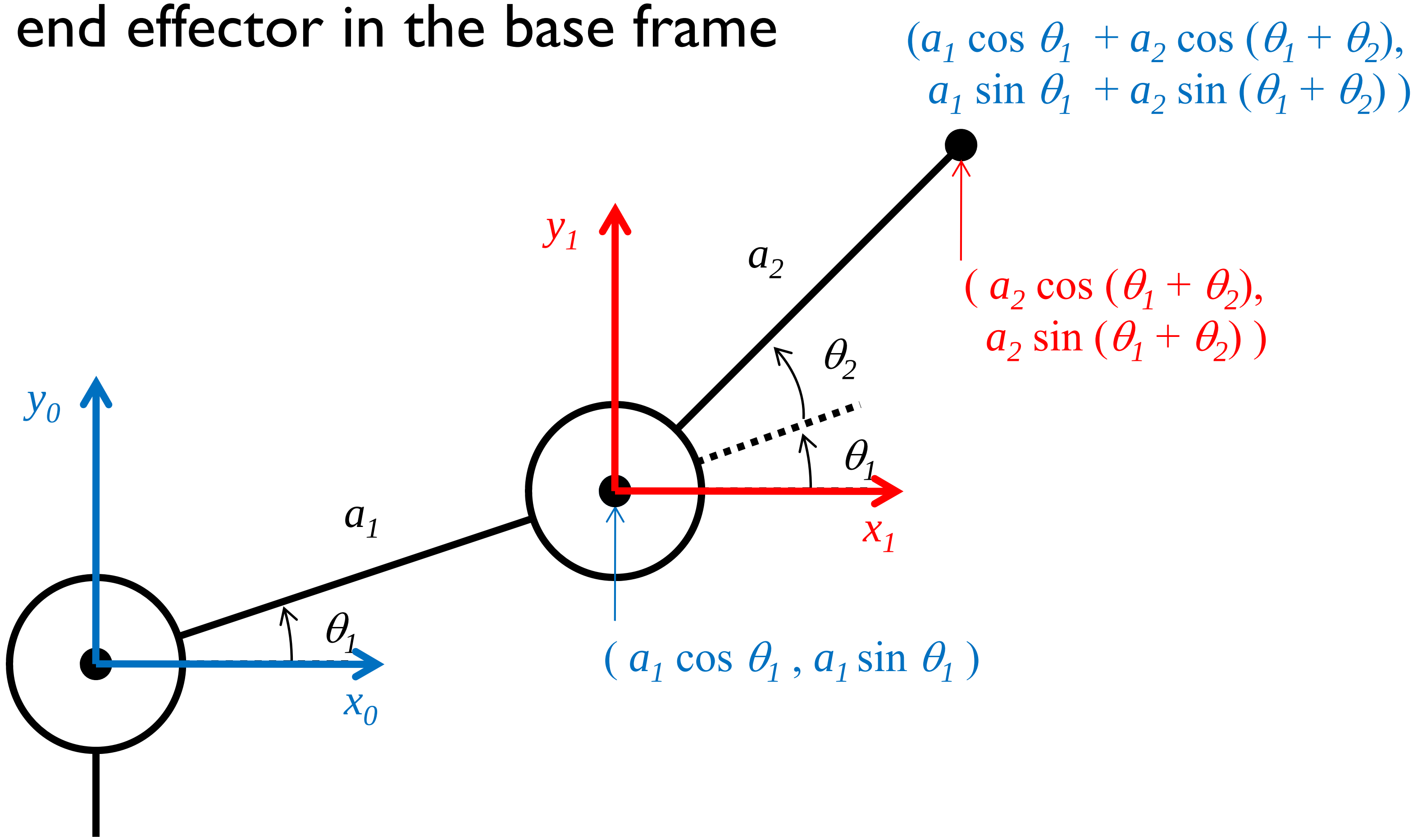
Forward Kinematics

- given the joint variables and dimensions of the links what is the position and orientation of the end effector?



Forward Kinematics

- because the base frame and frame *1* have the same orientation, we can sum the coordinates to find the position of the end effector in the base frame



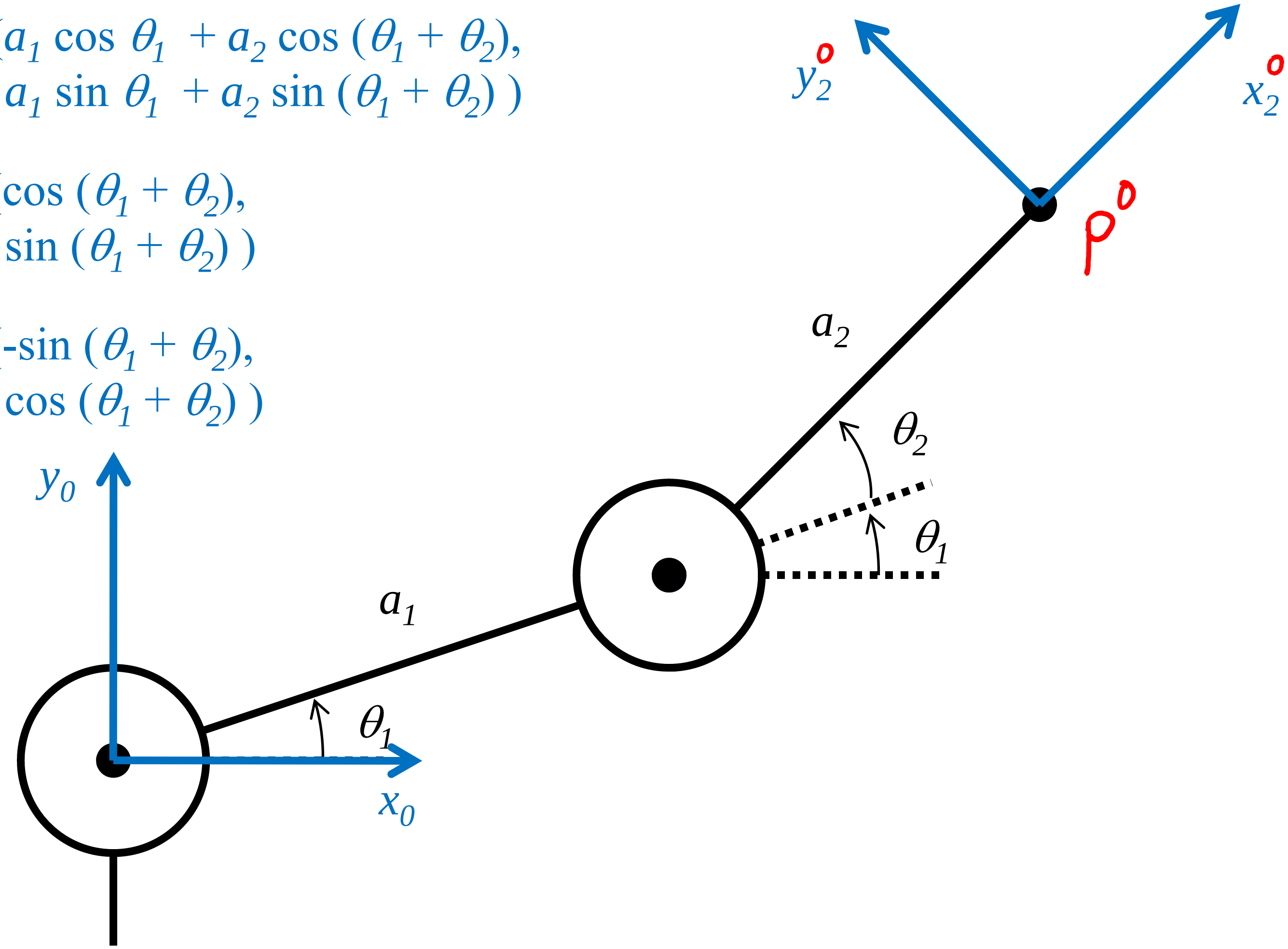
Forward Kinematics

- from earlier in the course

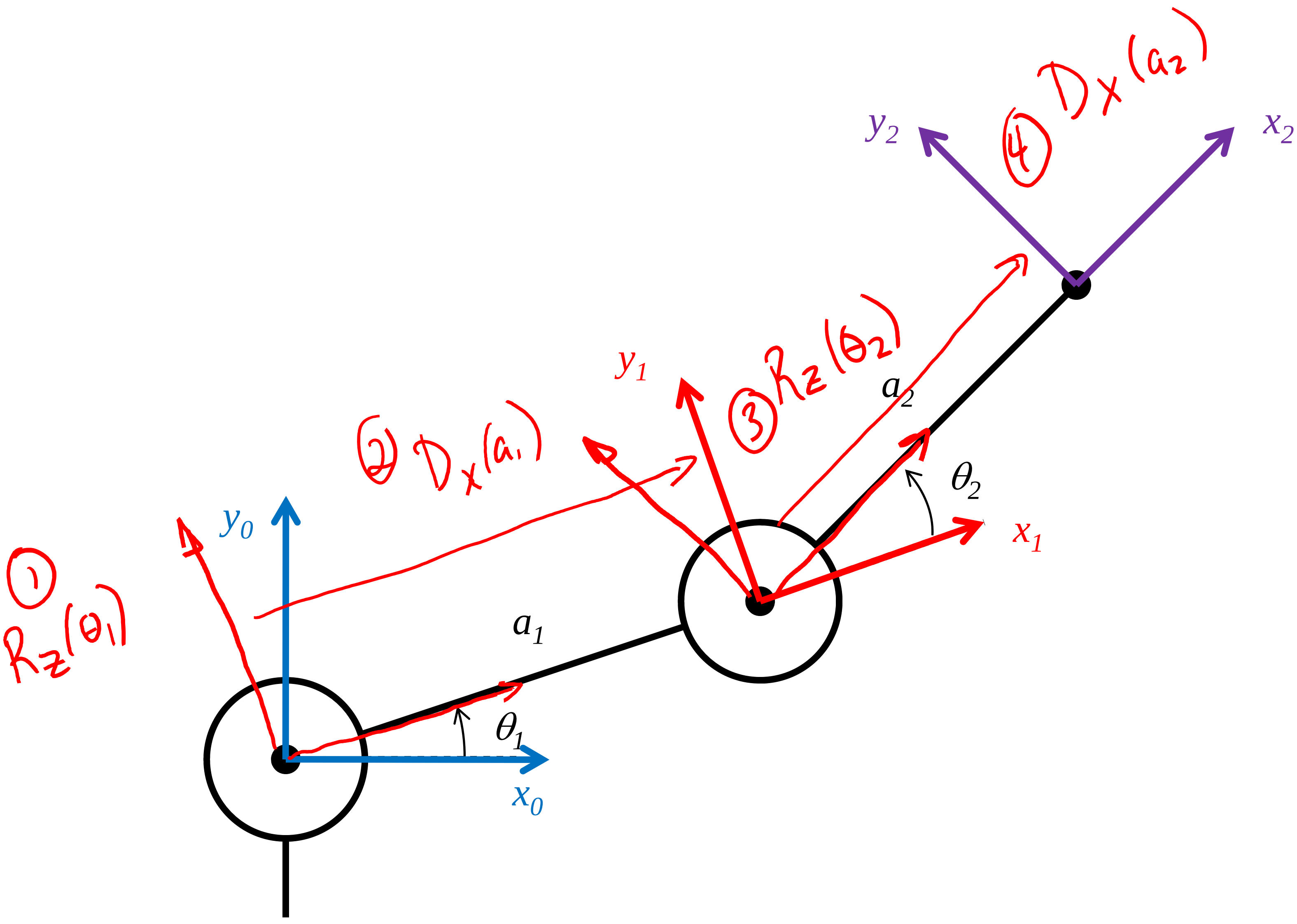
$$p^0 = (a_1 \cos \theta_1 + a_2 \cos (\theta_1 + \theta_2),\ a_1 \sin \theta_1 + a_2 \sin (\theta_1 + \theta_2))$$

$$x_2^0 = (\cos (\theta_1 + \theta_2),\ \sin (\theta_1 + \theta_2))$$

$$y_2^0 = (-\sin (\theta_1 + \theta_2),\ \cos (\theta_1 + \theta_2))$$



Frames



Forward Kinematics

- using transformation matrices

pose of {1} expressed in {0}

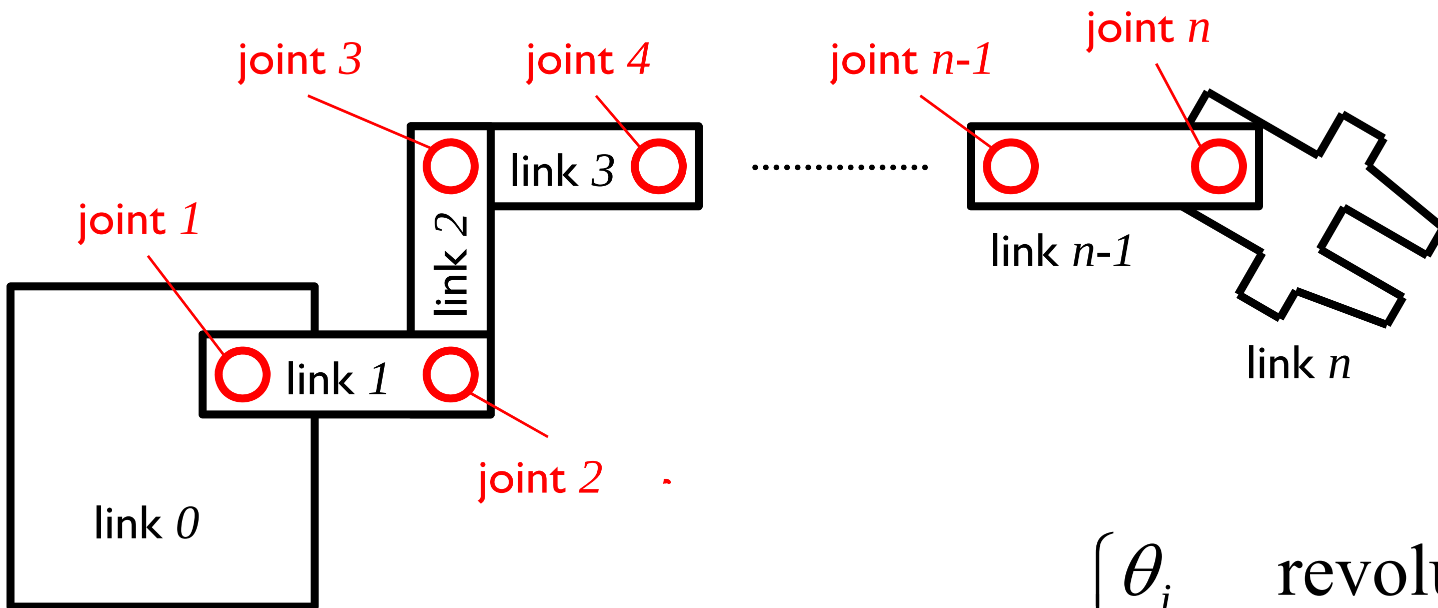
$$T_1^0 = R_{z,\theta_1} D_{x,a_1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = R_{z,\theta_2} D_{x,a_2}$$

$$T_2^0 = T_1^0 T_2^1$$

$$= \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Links and Joints



$$q_i = \begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$$

- n joints, $n + 1$ links
- link 0 is fixed (the base)
- joint i connects link $i - 1$ to link i
  - link i moves when joint i is actuated

Forward Kinematics

- attach a frame $\{i\}$ to link i
  - all points on link i are constant when expressed in $\{i\}$
  - if joint i is actuated then frame $\{i\}$ moves relative to frame $\{i - 1\}$
    - motion is described by the rigid transformation

$$T_i^{i-1}$$

    - the state of joint i is a function of its joint variable q_i (i.e., is a function of q_i)

$$T_i^{i-1} = T_i^{i-1}(q_i)$$

- this makes it easy to find the last frame with respect to the base frame

$$T_n^0 = T_1^0 \, T_2^1 \, T_3^2 \cdots T_n^{n-1}$$

Forward Kinematics

- more generally

$$T^i_j = \begin{cases} T^i_{i+1}\, T^{i+1}_{i+2} \ldots T^{j-1}_j & \text{if } \; i < j \\ I & \text{if } \; i = j \\ \left(T^i_j\right)^{-1} & \text{if } \; i > j \end{cases}$$

- the forward kinematics problem has been reduced to matrix multiplication

Forward Kinematics

- Denavit J and Hartenberg RS, "A kinematic notation for lower-pair mechanisms based on matrices." *Trans ASME J. Appl. Mech,* 23:215–221, 1955
  - described a convention for standardizing the attachment of frames on links of a serial linkage
- common convention for attaching reference frames on links of a serial manipulator and computing the transformations between frames

Denavit-Hartenberg

(origin) position of {1} expressed in {0}

o_1^0

$$T_i^{i-1} = R_{z,\theta_i} T_{z,d_i} T_{x,a_i} R_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a_i link length

α_i link twist

d_i link offset

θ_i joint angle

Denavit-Hartenberg

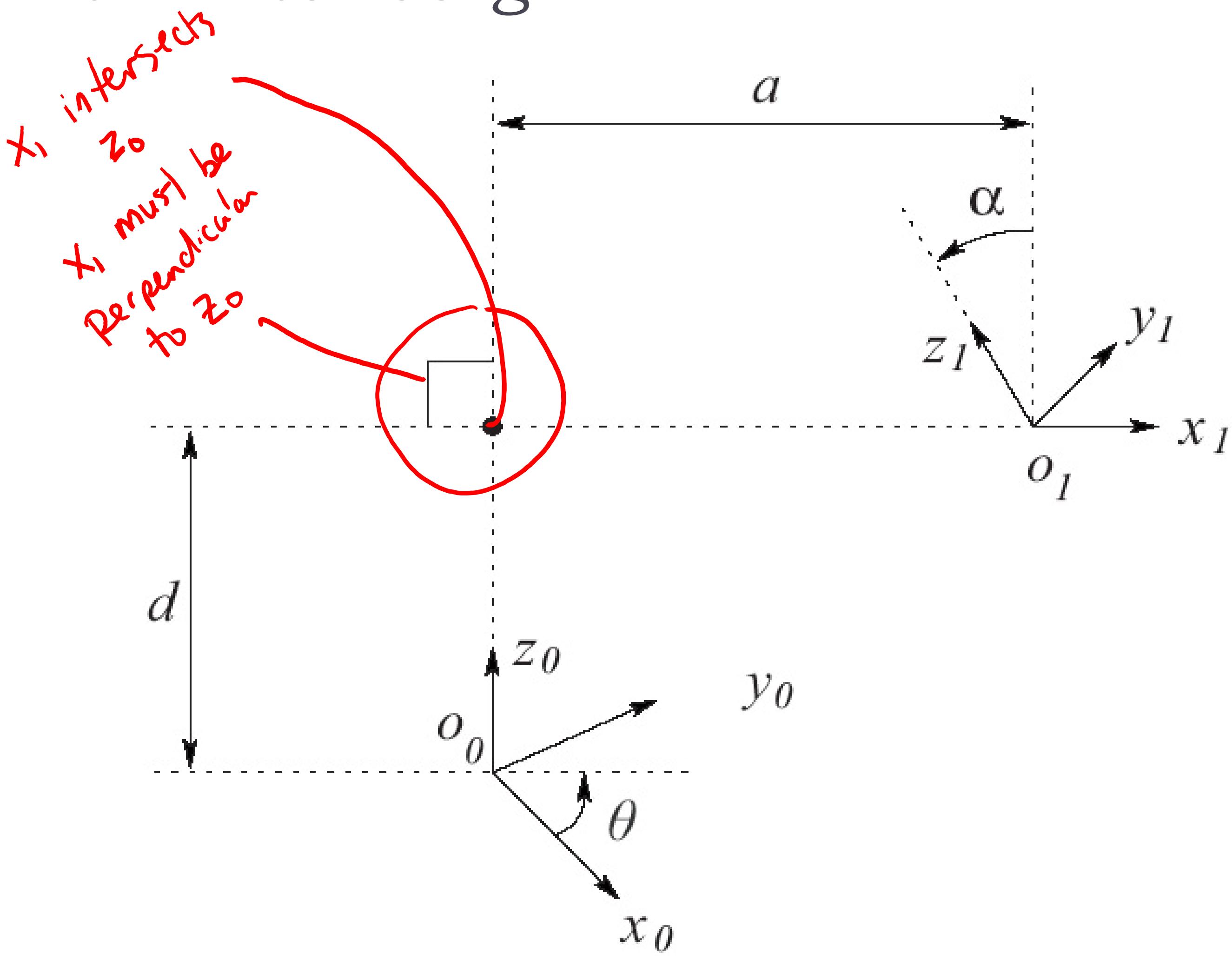


Figure 3.2: Coordinate frames satisfying assumptions DH1 and DH2.

Denavit-Hartenberg

- notice the form of the rotation component

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} \end{bmatrix}$$

  - this does not look like it can represent arbitrary rotations
- can the DH convention actually describe every physically possible link configuration?

Denavit-Hartenberg

- yes, but we must choose the orientation and position of the frames in a certain way

  - (DH1) $\hat{x}_i \perp \hat{z}_{i-1}$

  - (DH2) $\hat{x}_i$ intersects $\hat{z}_{i-1}$

- claim: if DH1 and DH2 are true then there exists unique numbers

$$a, d, \theta, \alpha \text{ such that } T_1^0 = R_{z,\theta}\, D_{z,d}\, D_{x,a}\, R_{x,\alpha}$$

Denavit-Hartenberg

- proof: on blackboard in class

rotation

$\hat{x}_i \perp \hat{z}_{i-1}$: $\hat{x}_i \cdot \hat{z}_{i-1} = 0$

$$\begin{bmatrix} r_{11} & & \\ r_{21} & & \\ 0 & r_{32} & r_{33} \end{bmatrix}$$

$= \hat{x}_i \cdot \hat{z}_{i-1}$

column 1 has magnitude equal to 1 $\Rightarrow r_{11}^2 + r_{21}^2 = 1$

$\therefore$ there exists some θ so that

$r_{11} = \cos\theta$

$r_{21} = \sin\theta$

row 3 has magnitude $= 1 \Rightarrow r_{32}^2 + r_{33}^2 = 1$

$\therefore$ there exists some α so that

$r_{32} = \sin\alpha$

$r_{33} = \cos\alpha$

translation

$$o_1^0 = o_0^0 + d\,\hat{z}_0^0 + a\,\hat{x}_1^0$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + a \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} a \cdot \cos\theta \\ a \cdot \sin\theta \\ d \end{bmatrix}$$

Denavit-Hartenberg

DH Parameters

- a_i : link length
  - distance between z_{i-1} and z_i measured along x_i
- α_i : link twist
  - angle between z_{i-1} and z_i measured about x_i
- d_i : link offset
  - distance between o_{i-1} to the intersection of x_i and z_{i-1} measured along z_{i-1}
- θ_i : joint angle
  - angle between x_{i-1} and x_i measured about z_{i-1}

Example with Frames Already Placed

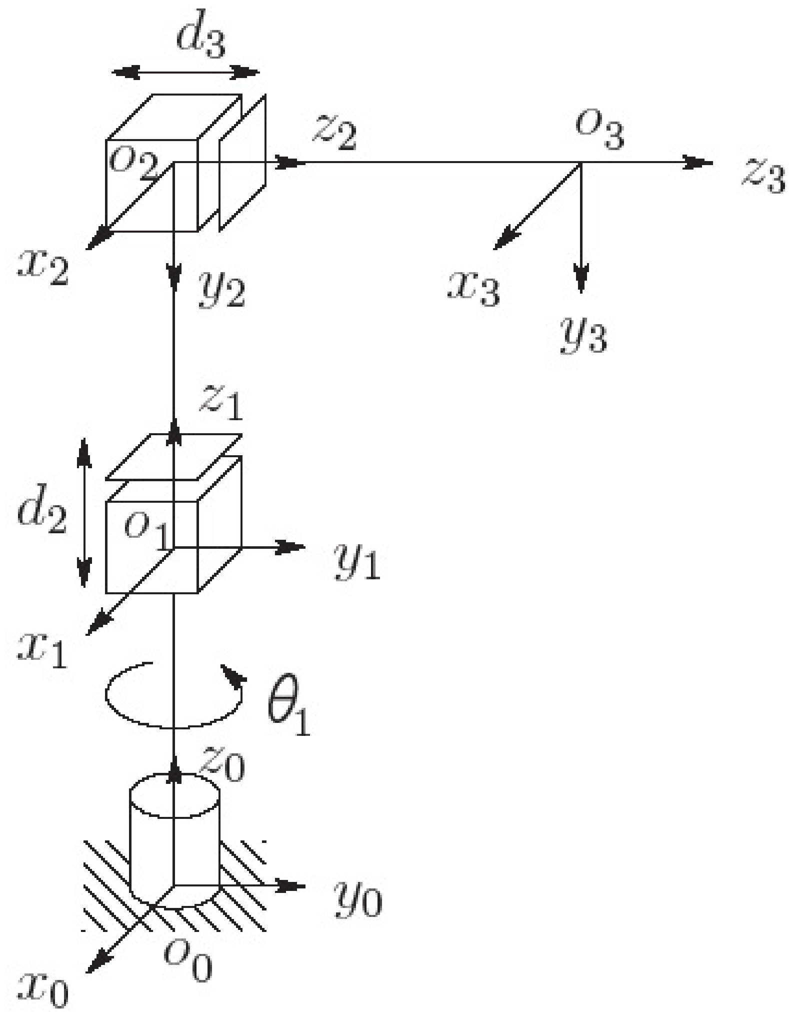


Figure 3.7: Three-link cylindrical manipulator.

Step 5: Find the DH parameters

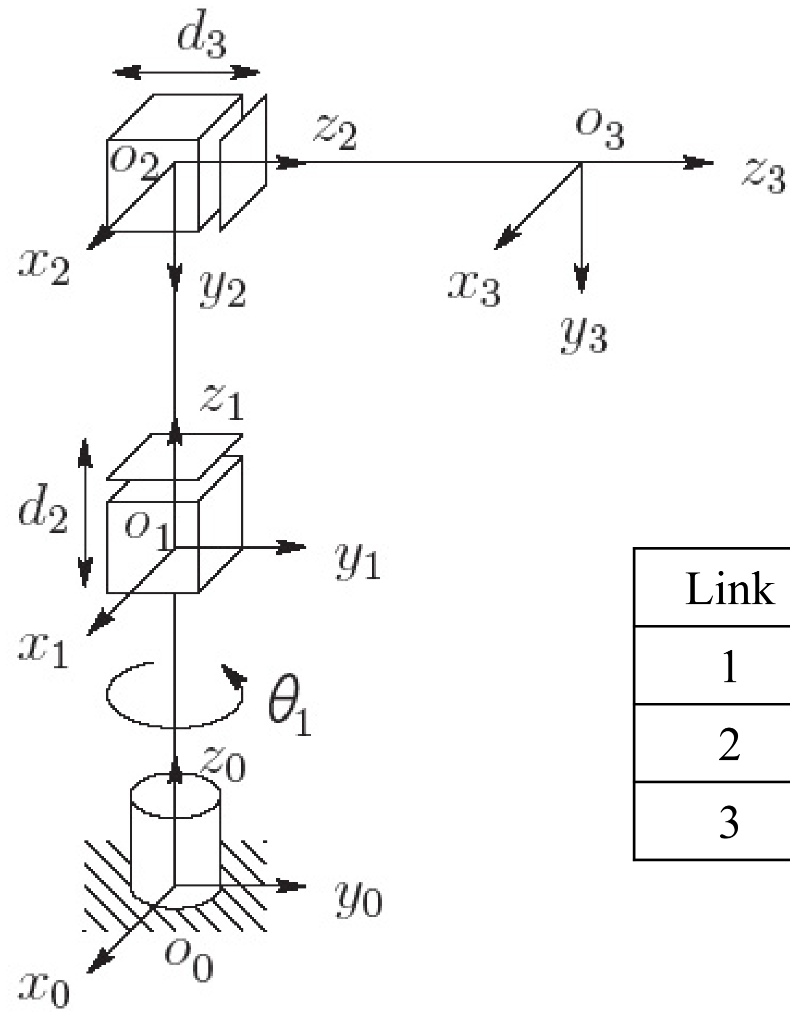


Figure 3.7: Three-link cylindrical manipulator.

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1*
2	0	-90	d_2*	0
3	0	0	d_3*	0

* joint variable

Denavit-Hartenberg Forward Kinematics

- RPP cylindrical manipulator
  - http://strobotics.com/cylindrical-format-robot.htm

Denavit-Hartenberg Forward Kinematics

How do we place the frames?

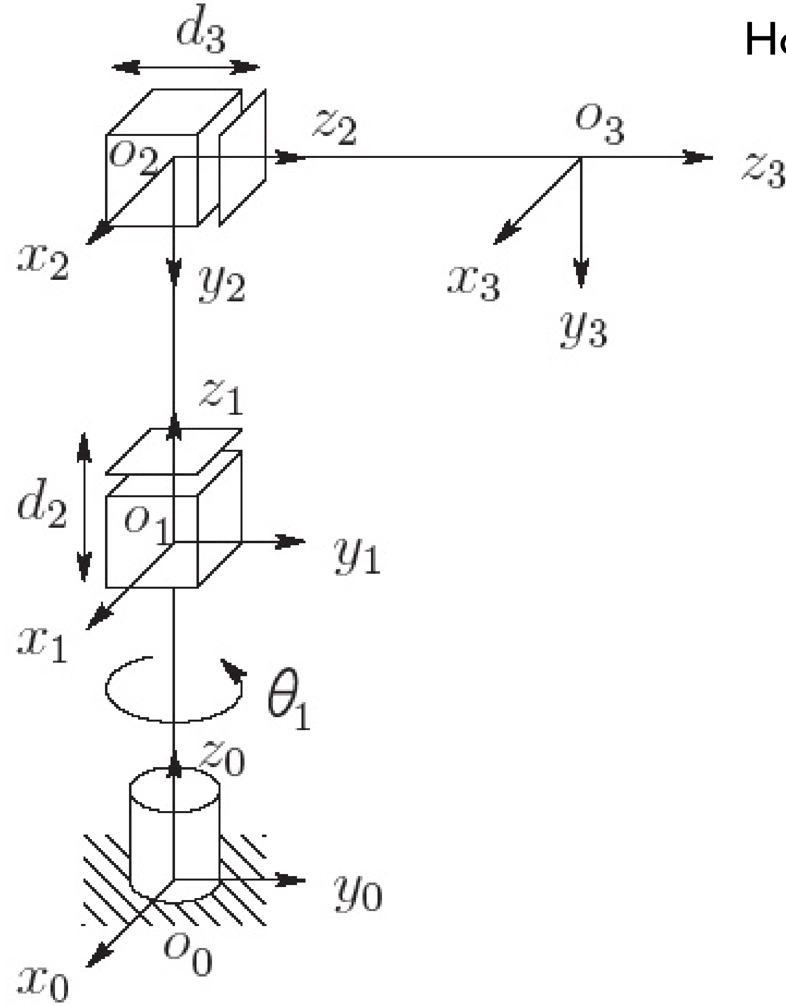


Figure 3.7: Three-link cylindrical manipulator.

Step 1: Choose the *z*-axis for each frame

- recall the DH transformation matrix

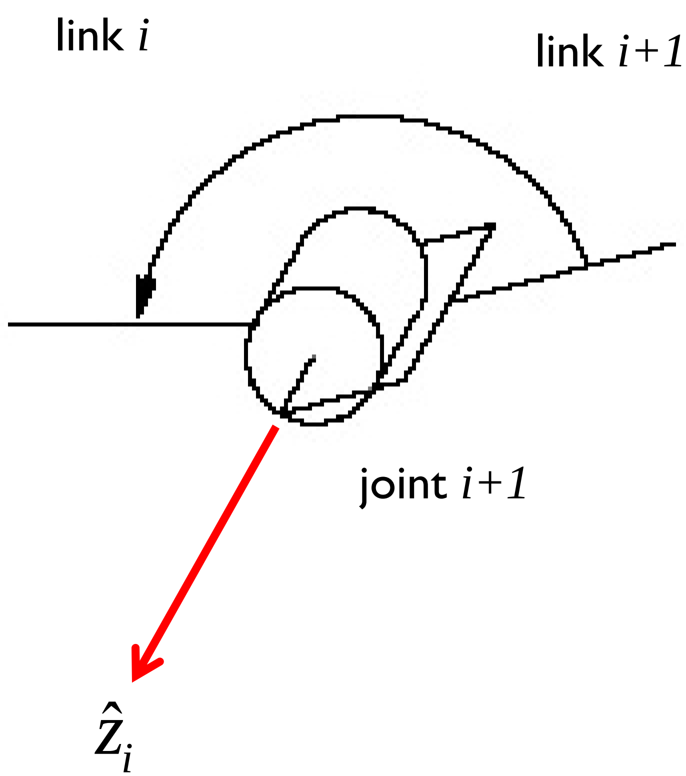
$$T_i^{i-1} = R_{z,\theta_i} T_{z,d_i} T_{x,a_i} R_{x,\alpha_i}$$

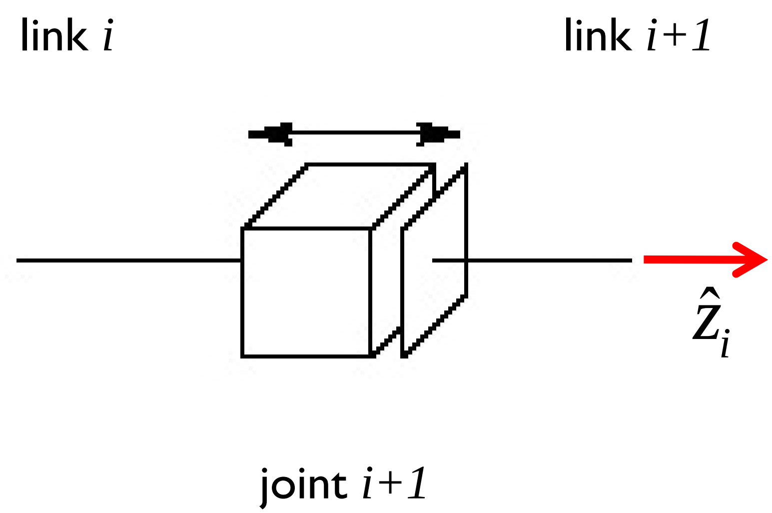
$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\hat{x}_i^{i-1}$ $\hat{y}_i^{i-1}$ $\hat{z}_i^{i-1}$

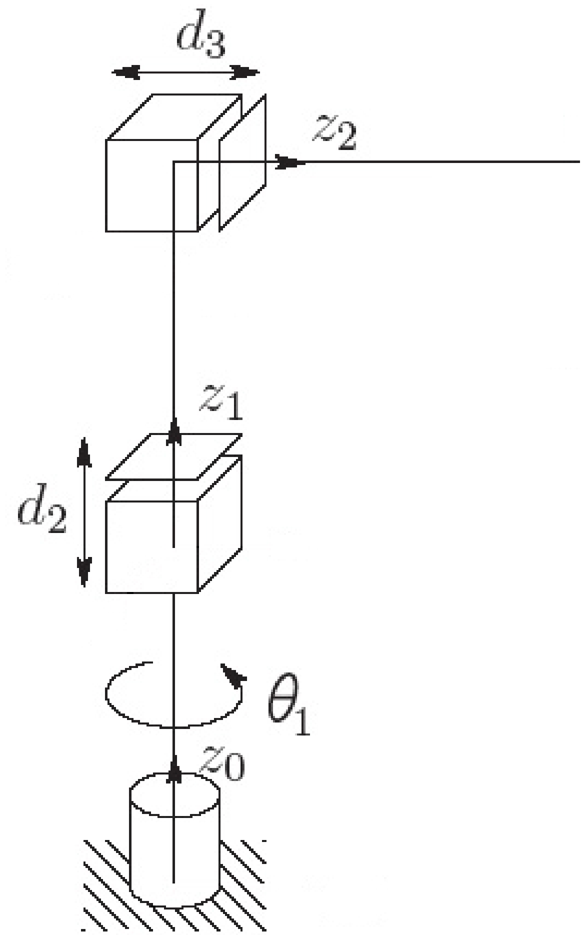
Step 1: Choose the z-axis for each frame

- $\hat{z}_i \equiv$ axis of actuation for joint $i+1$





Step 1: Choose the z-axis for each frame

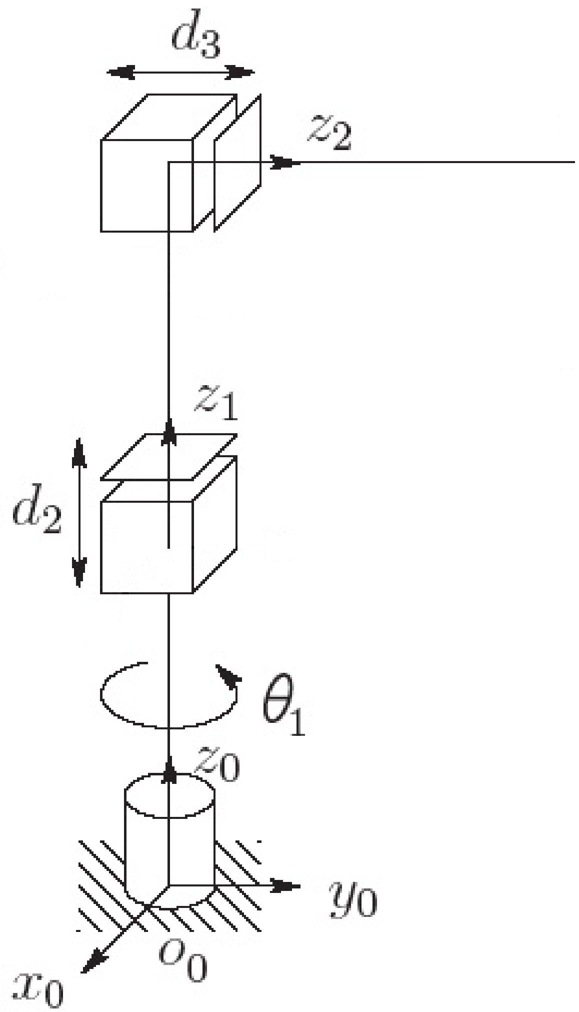


- Warning: the picture is deceiving. We do not yet know the origin of the frames; all we know at this point is that each z_i points along a joint axis

Step 2: Establish frame {0}

- place the origin o_0 anywhere on z_0
  - often the choice of location is obvious
- choose x_0 and y_0 so that $\{0\}$ is right-handed
  - often the choice of directions is obvious

Step 2: Establish frame {0}

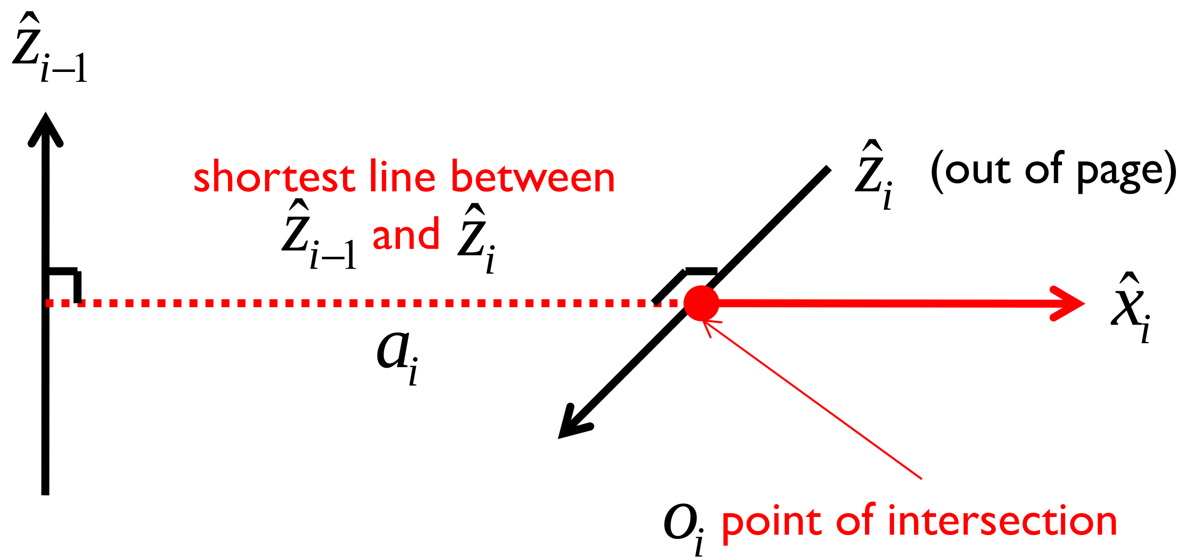


Step 3: Iteratively construct {1}, {2}, … {n-1}

- using frame {i-1} construct frame {i}
  - DH1: x_i is perpendicular to z_{i-1}
  - DH2: x_i intersects z_{i-1}
- 3 cases to consider depending on the relationship between z_{i-1} and z_i

Step 3: Iteratively construct {1}, {2}, … {n-1}

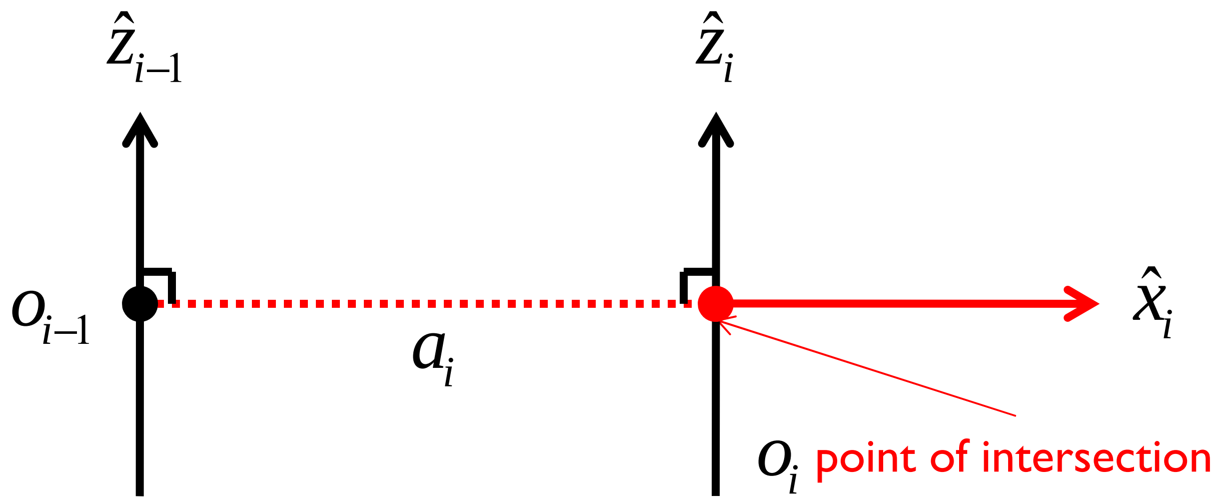
- Case 1
  - z_{i-1} and z_i are not coplanar (skew)



  - α_i angle from z_{i-1} to z_i measured about x_i

Step 3: Iteratively construct {1}, {2}, … {n-1}

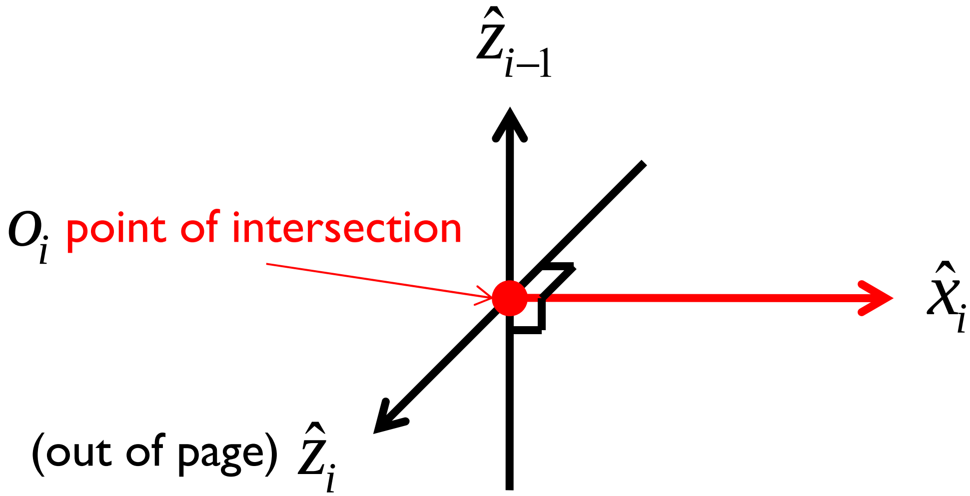
- Case 2
  - z_{i-1} and z_i are parallel ($\alpha_i = 0$)



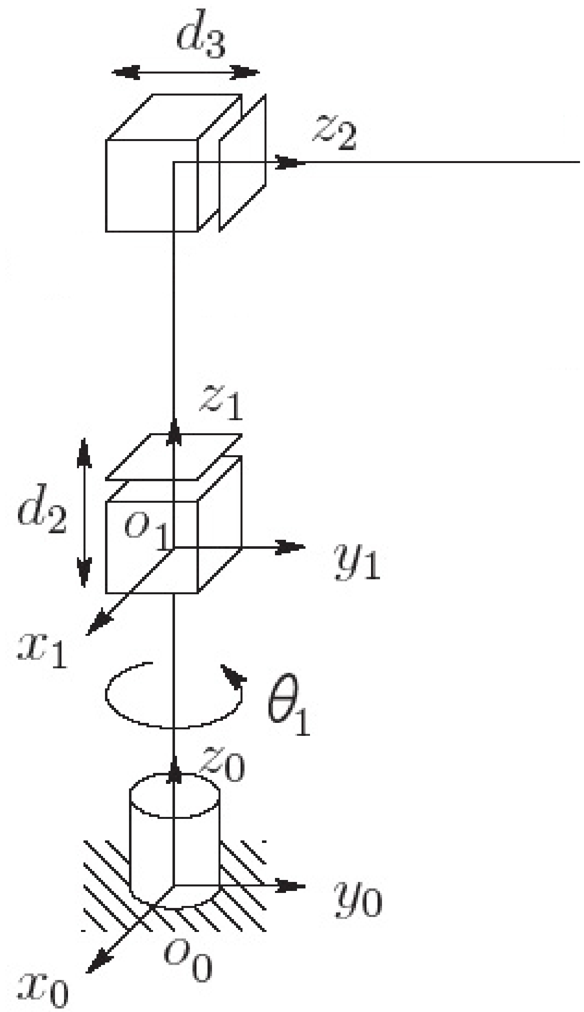
  - notice that this choice results in $d_i = 0$

Step 3: Iteratively construct {1}, {2}, … {n-1}

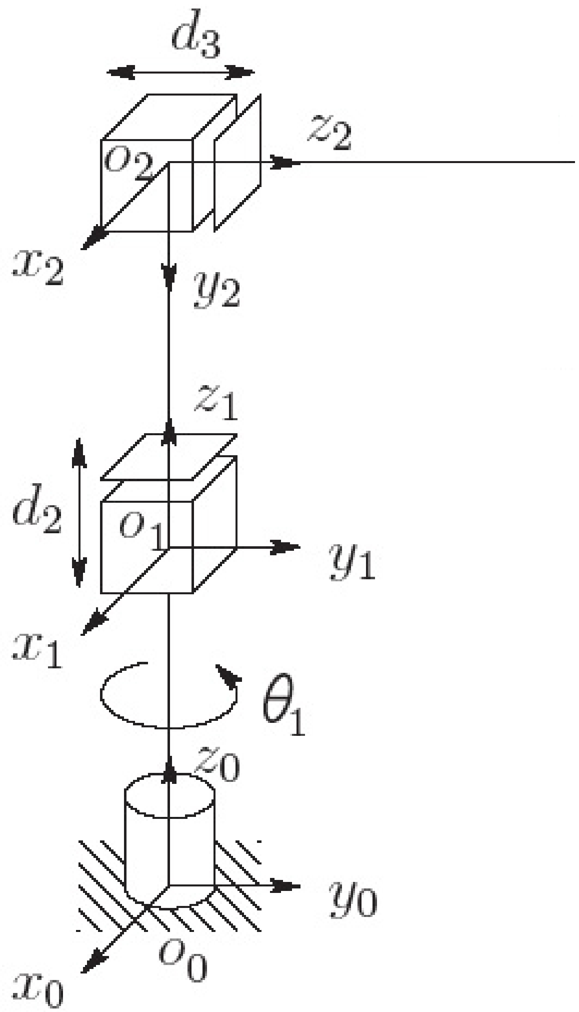
- Case 3
  - z_{i-1} and z_i intersect ($a_i = 0$)



Step 3: Iteratively construct {1}, {2}, ... {*n*-1}



Step 3: Iteratively construct {1}, {2}, ... {*n*-1}



Step 4: Place the end effector frame

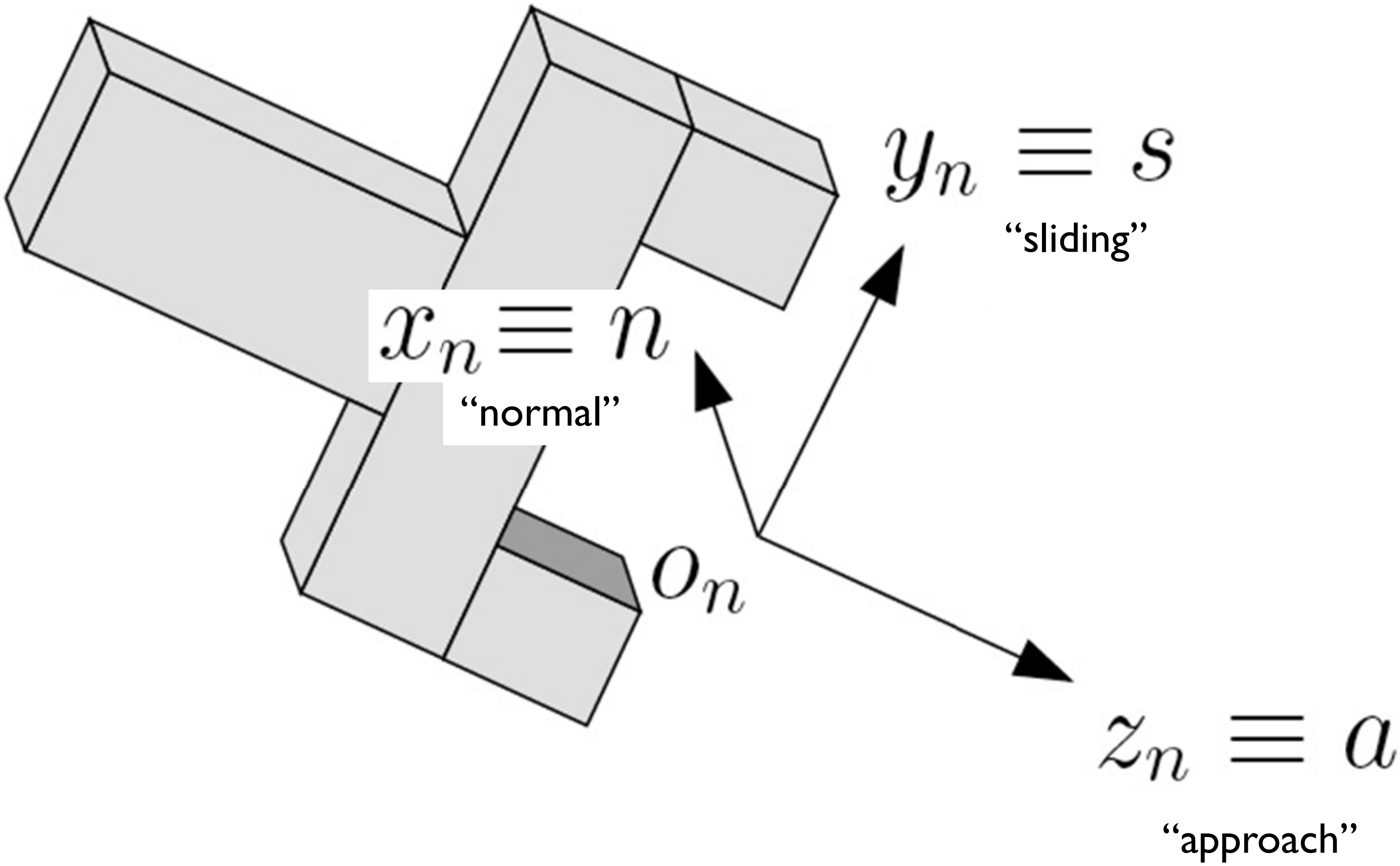


Figure 3.5: Tool frame assignment.

Step 4: Place the end effector frame

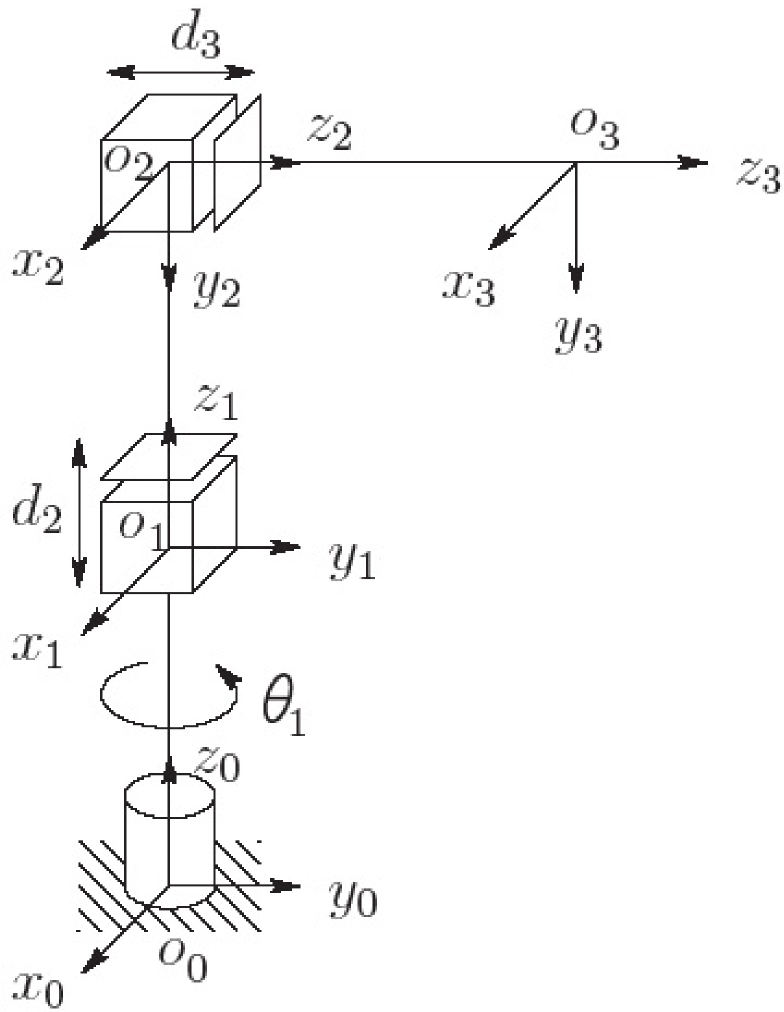


Figure 3.7: Three-link cylindrical manipulator.

Step 5: Find the DH parameters

- a_i : distance between z_{i-1} and z_i measured along x_i
- α_i : angle between z_{i-1} and z_i measured about x_i
- d_i : distance between o_{i-1} to the intersection of x_i and z_{i-1} measured along z_{i-1}
- θ_i : angle between x_{i-1} and x_i measured about z_{i-1}

Step 5: Find the DH parameters

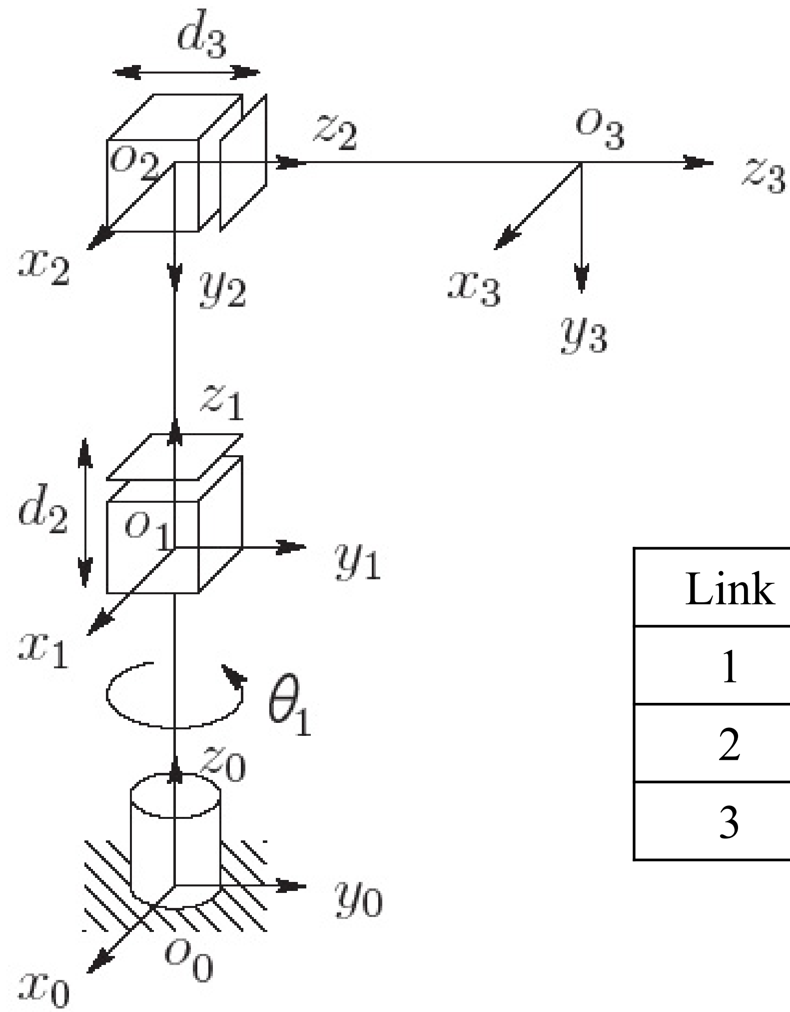


Figure 3.7: Three-link cylindrical manipulator.

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1*
2	0	-90	d_2*	0
3	0	0	d_3*	0

* joint variable

More Denavit-Hartenberg Examples

Step 5: Find the DH parameters

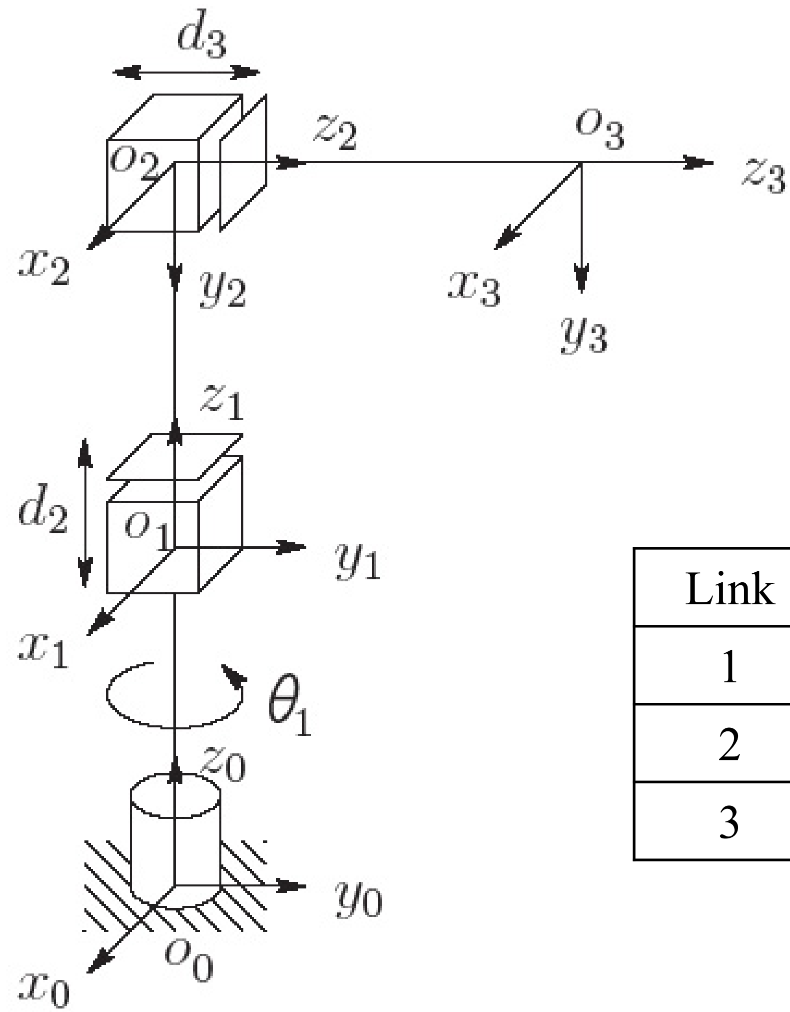


Figure 3.7: Three-link cylindrical manipulator.

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1*
2	0	-90	d_2*	0
3	0	0	d_3*	0

* joint variable

Step 6: Compute the transformation

- once the DH parameters are known, it is easy to construct the overall transformation

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1*
2	0	-90	d_2*	0
3	0	0	d_3*	0

* joint variable

$$T_1^0 = R_{z,\theta_1} T_{z,d_1} T_{x,a_1} R_{x,\alpha_1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 6: Compute the transformation

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1*
2	0	-90	d_2*	0
3	0	0	d_3*	0

* joint variable

$$T_2^1 = R_{z,\theta_2} T_{z,d_2} T_{x,a_2} R_{x,\alpha_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 6: Compute the transformation

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1*
2	0	-90	d_2*	0
3	0	0	d_3*	0

* joint variable

$$T_3^2 = R_{z,\theta_3} T_{z,d_3} T_{x,a_3} R_{x,\alpha_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 6: Compute the transformation

$$T_3^0 = T_1^0 T_2^1 T_3^2 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Spherical Wrist

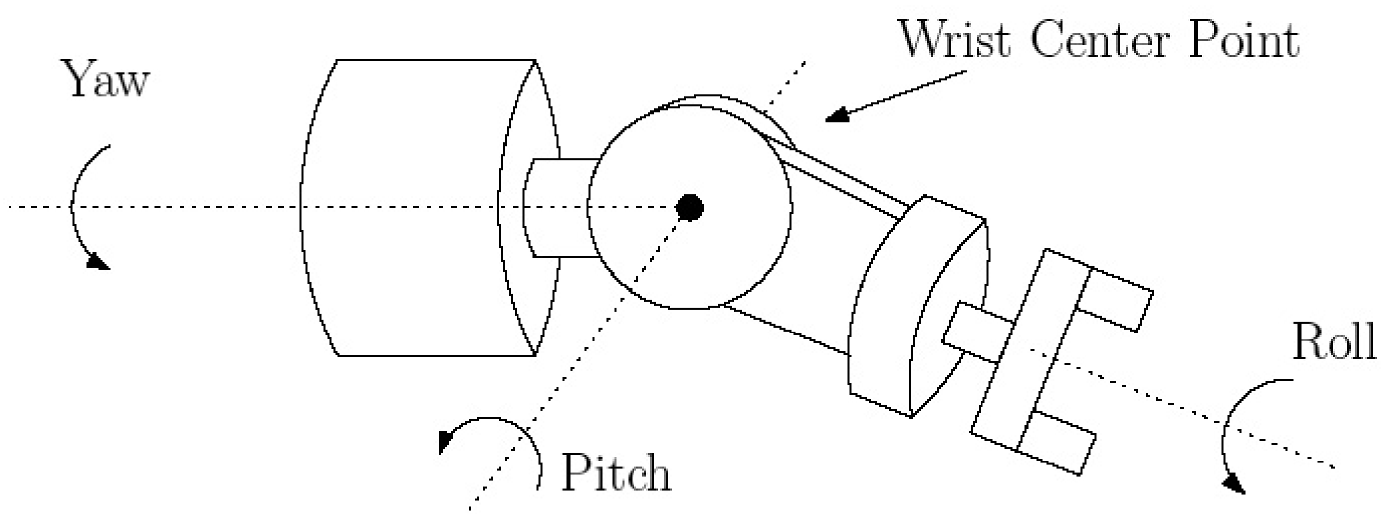
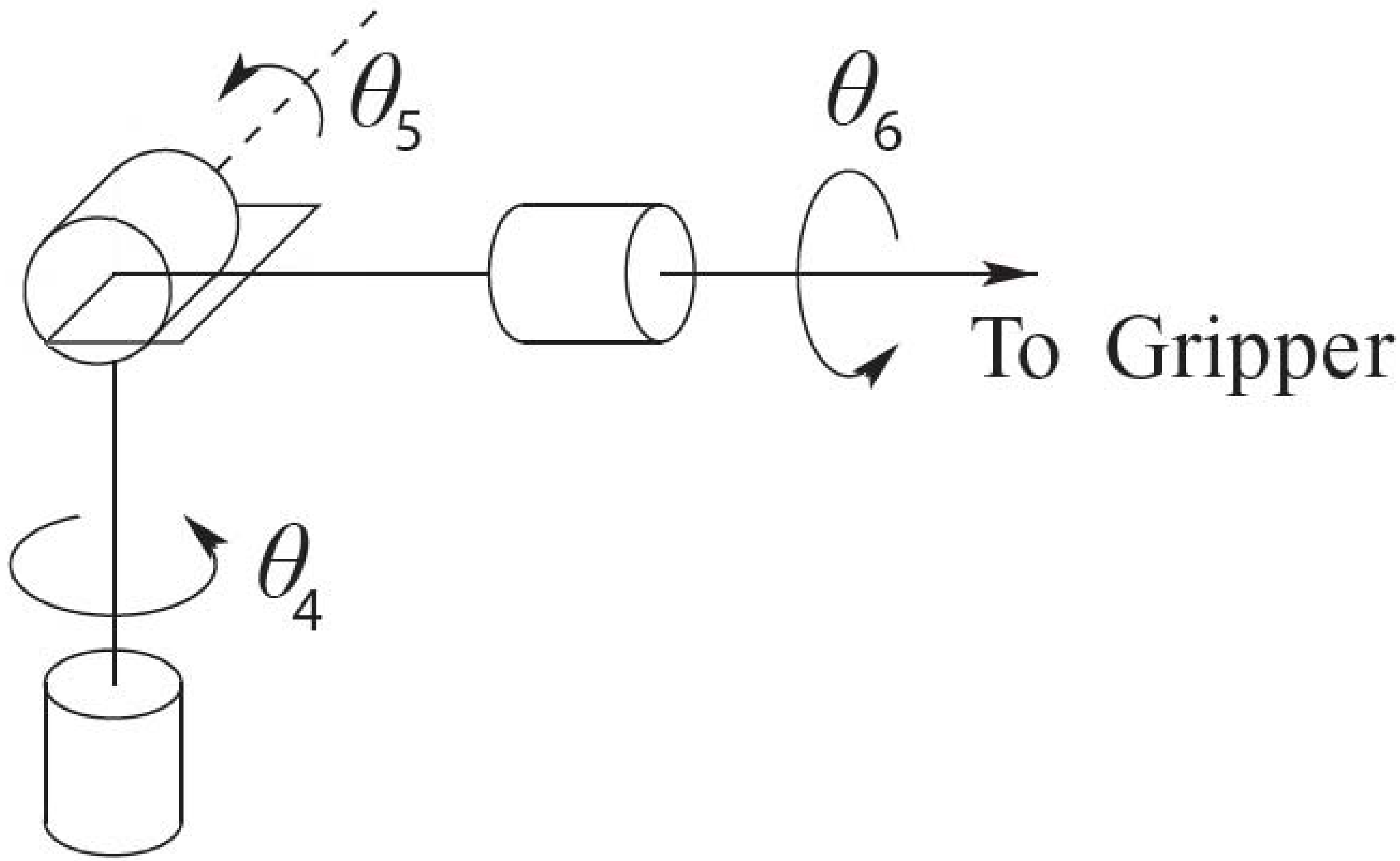
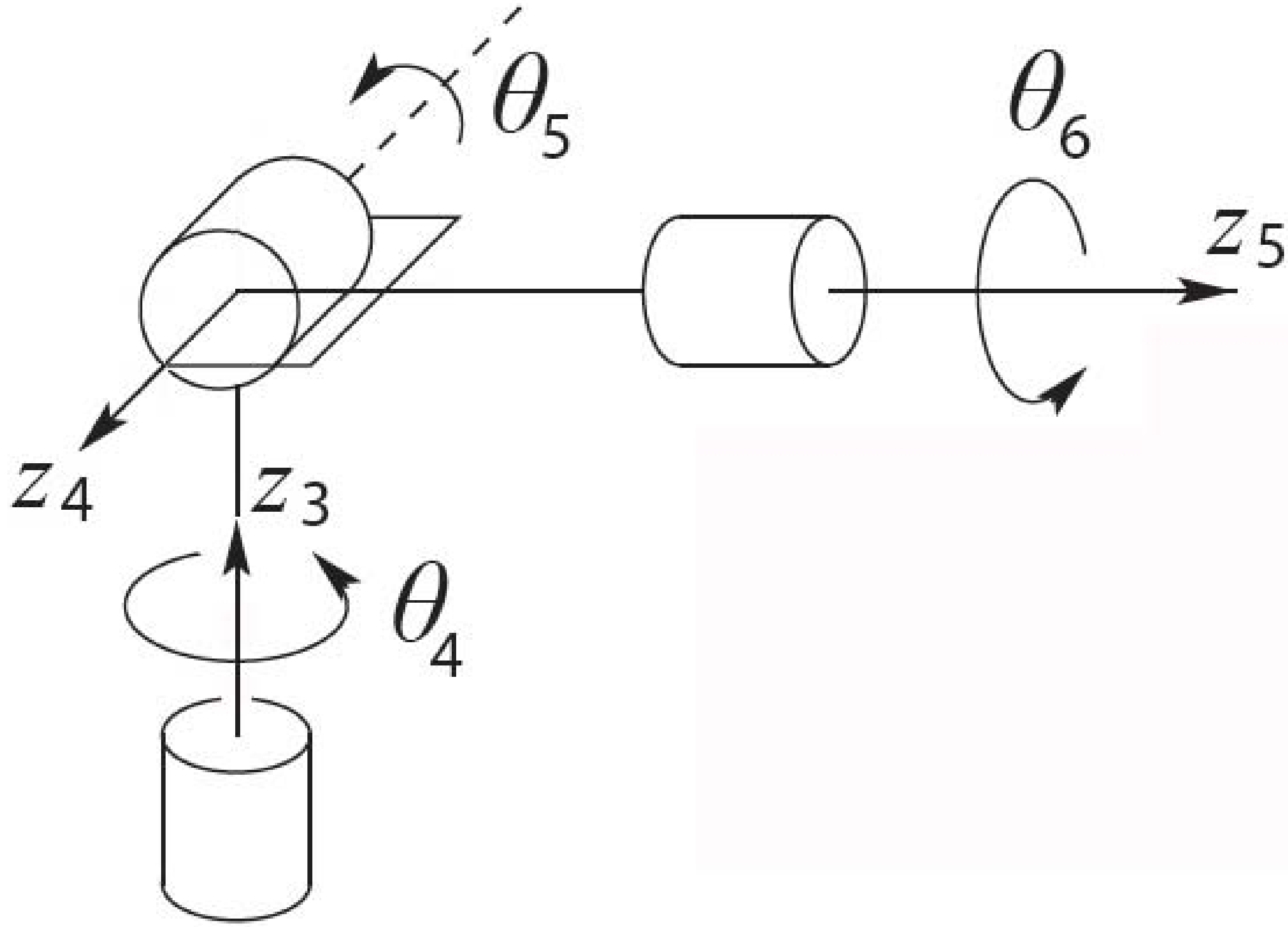


Figure 1.6: The spherical wrist. The axes of rotation of the spherical wrist are typically denoted roll, pitch, and yaw and intersect at a point called the wrist center point.

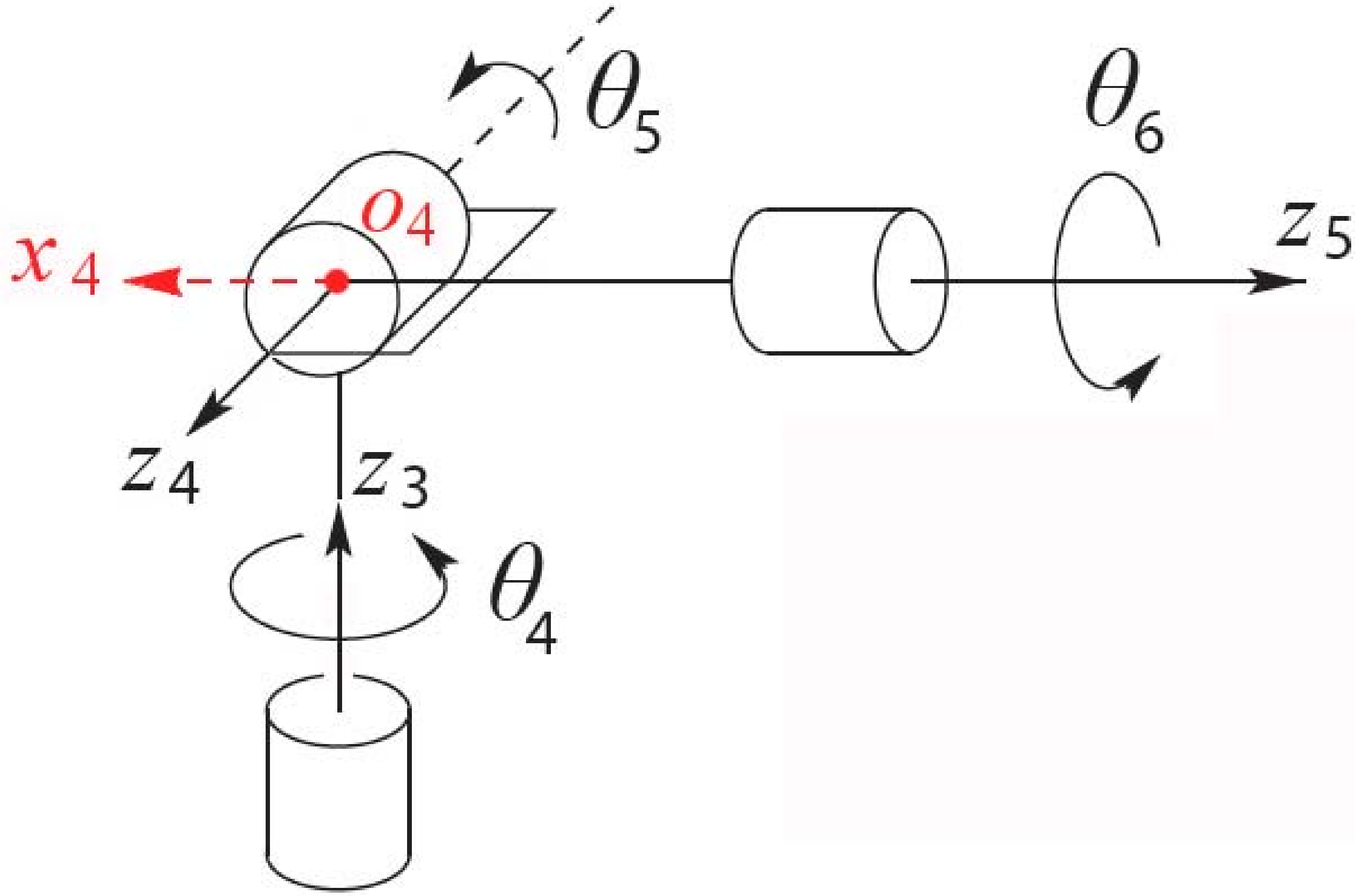
Spherical Wrist



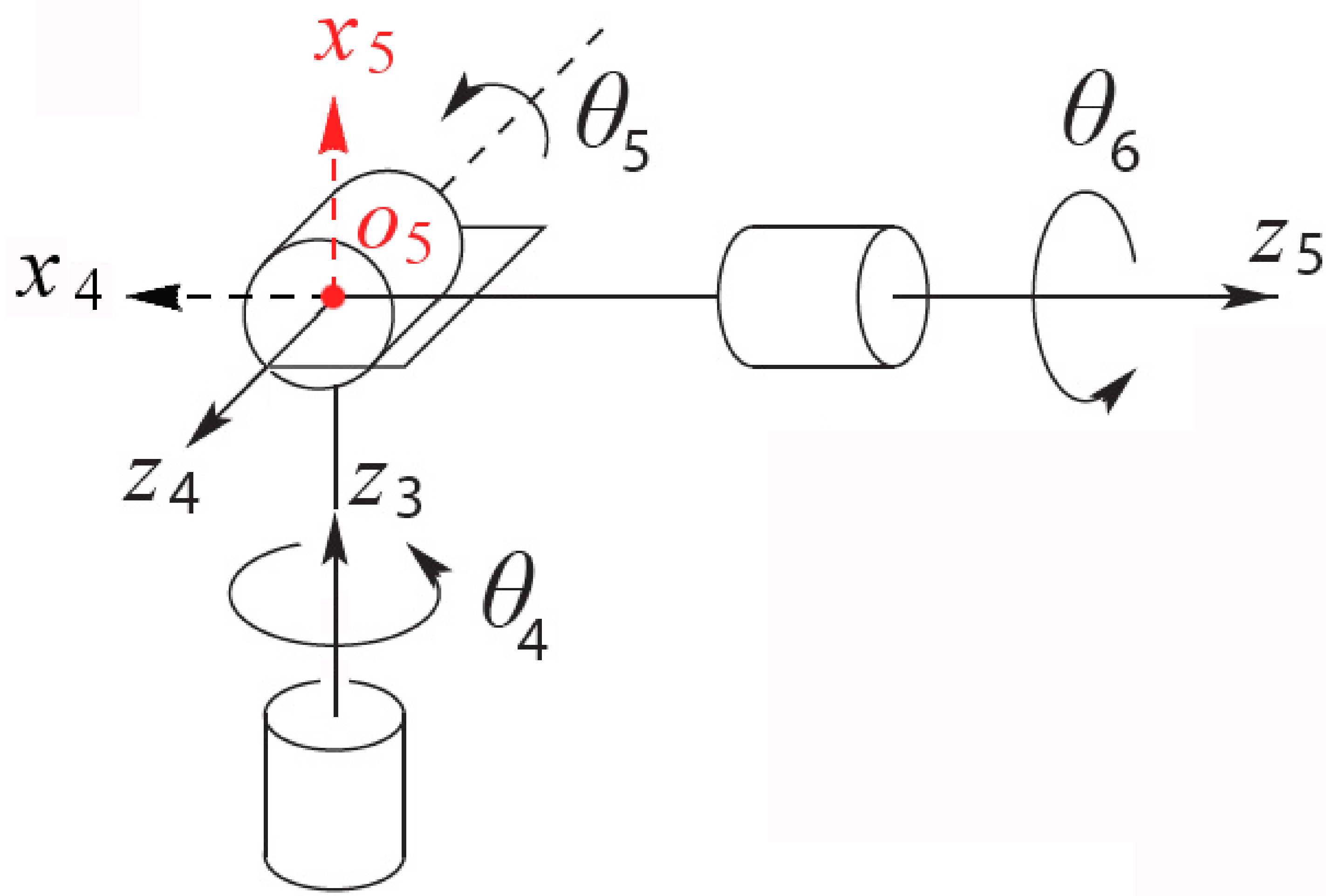
Spherical Wrist: Step 1



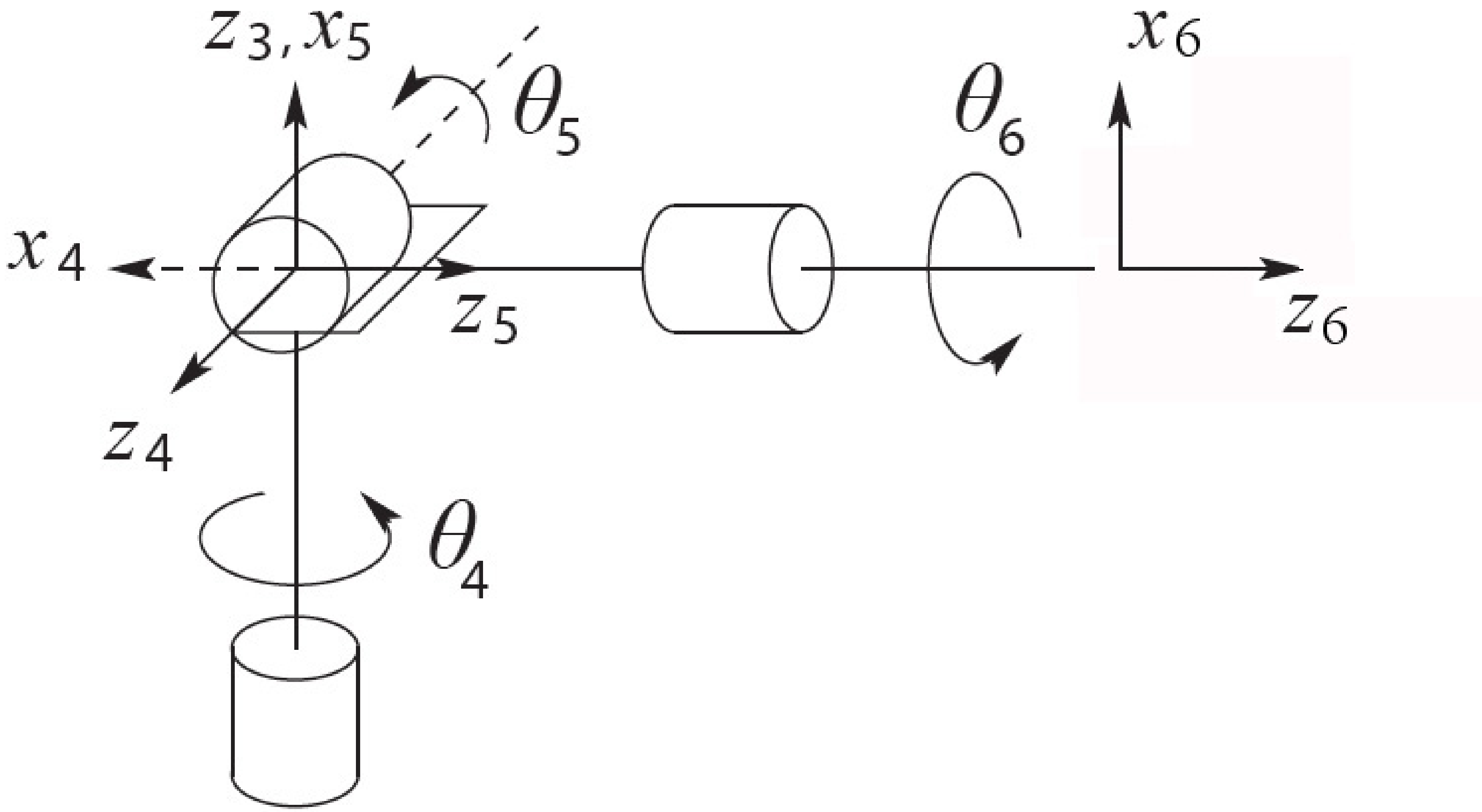
Spherical Wrist: Step 2



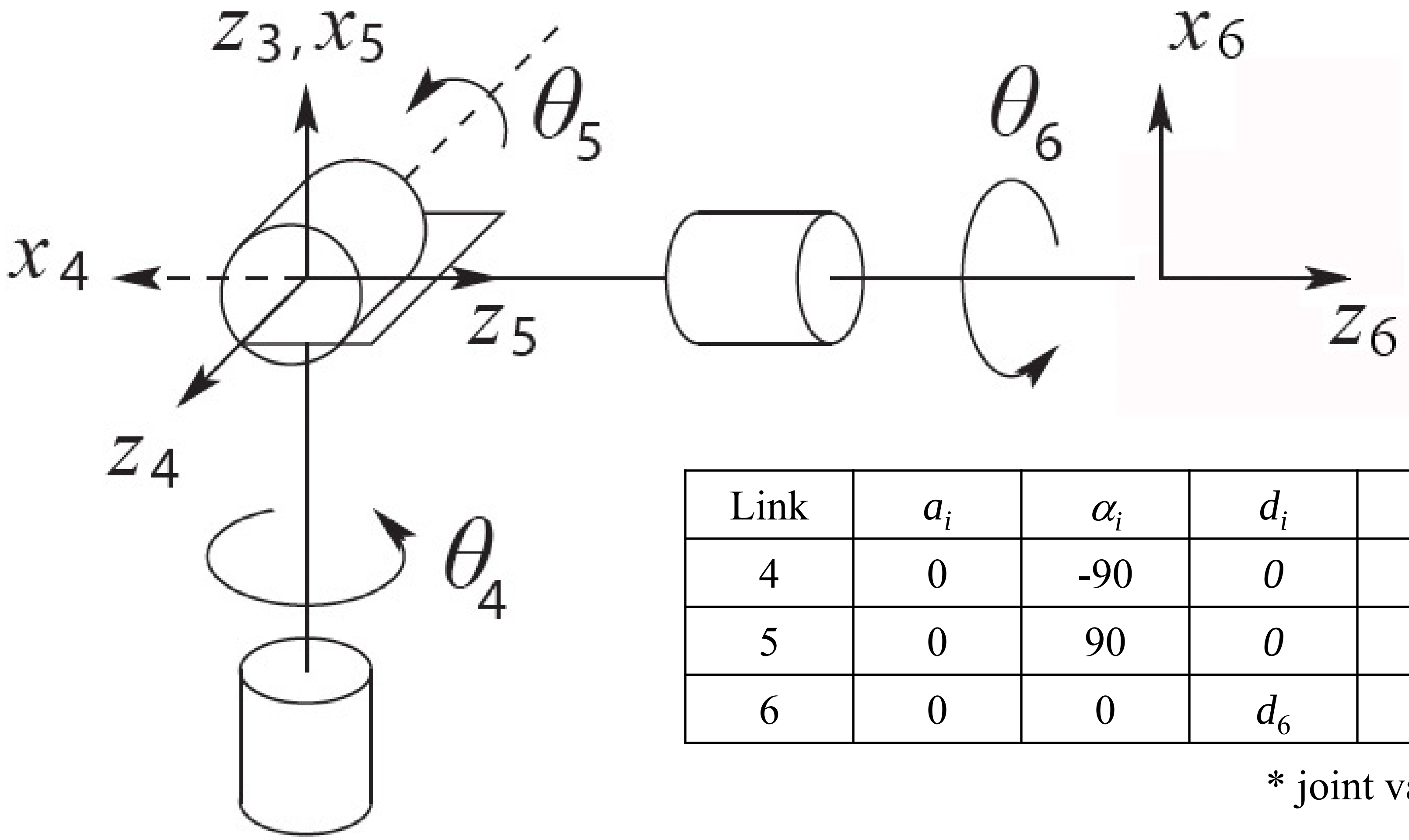
Spherical Wrist: Step 2



Spherical Wrist: Step 4



Step 5: DH Parameters



Link	a_i	α_i	d_i	θ_i
4	0	-90	*0*	θ_4*
5	0	90	*0*	θ_5*
6	0	0	d_6	θ_6*

* joint variable

Step 6: Compute the transformation

$$T_6^3 = T_4^3 T_5^4 T_6^5 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

RPP + Spherical Wrist

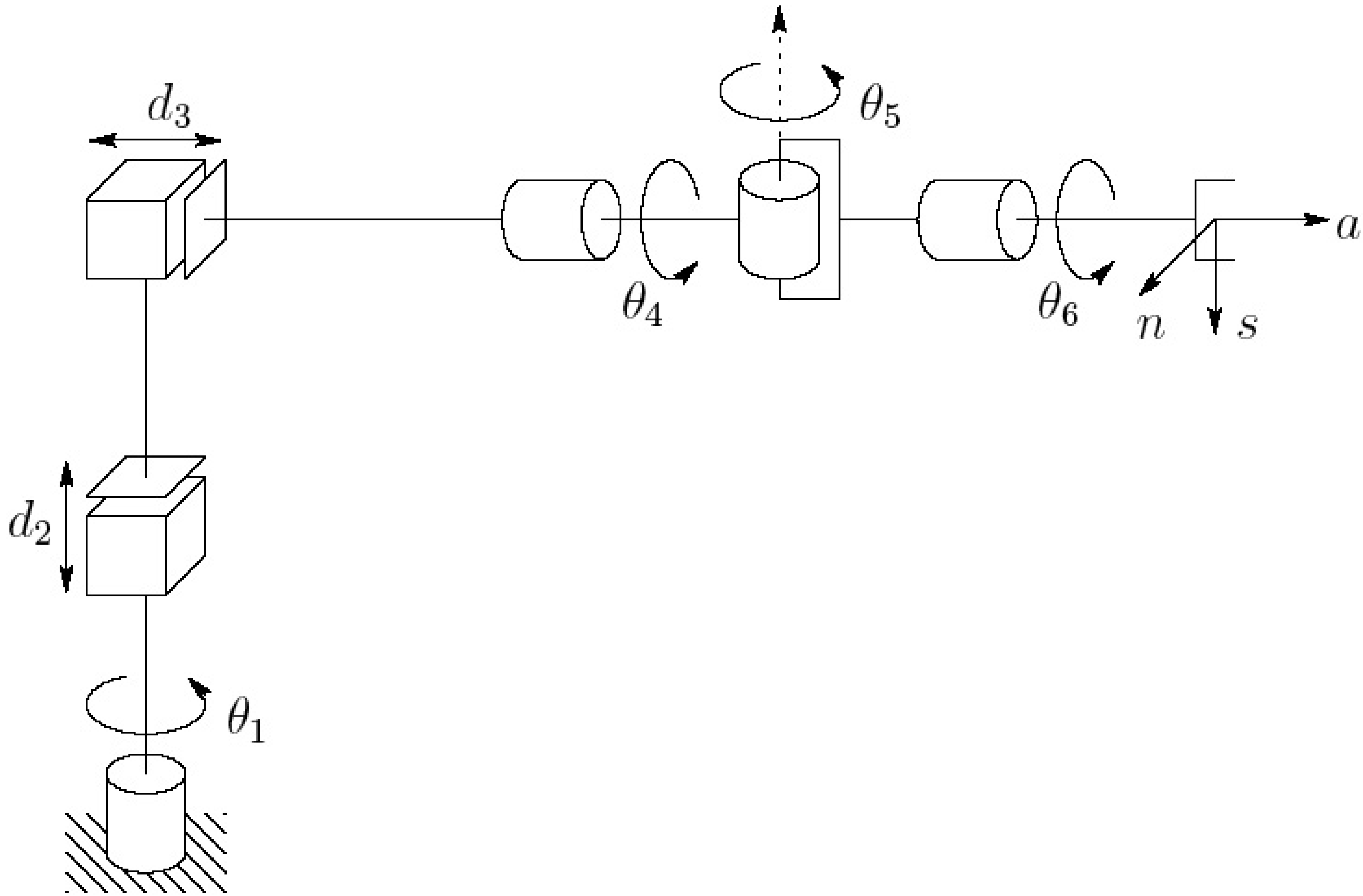


Figure 3.9: Cylindrical robot with spherical wrist.

RPP + Spherical Wrist

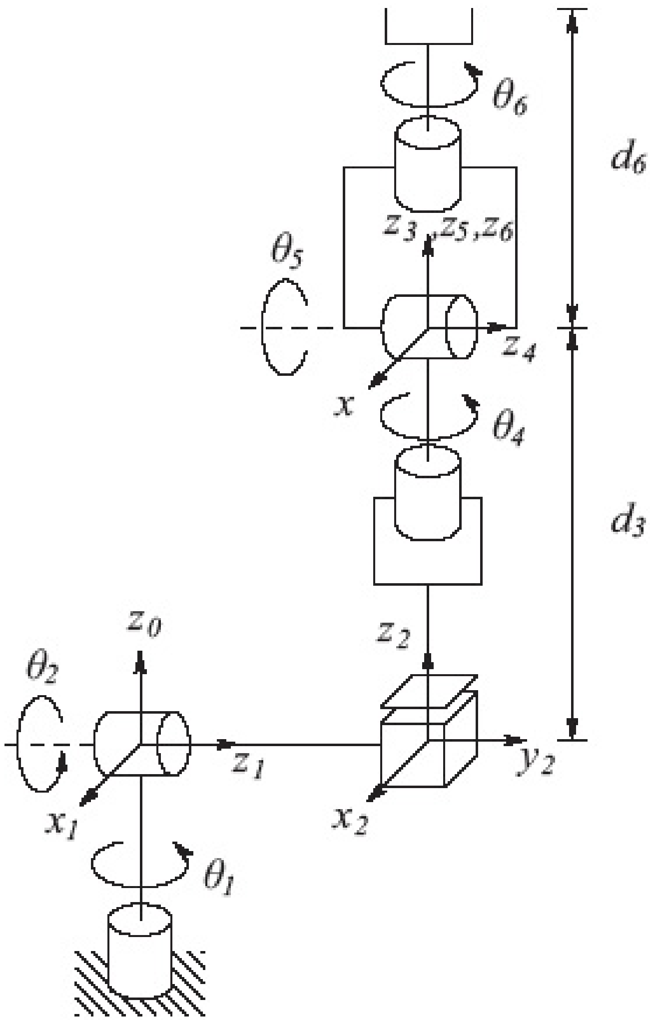
$$T_6^0 = T_3^0 T_6^3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{11} = c_1 c_4 c_5 c_6 - c_1 s_4 s_6 + s_1 s_5 c_6$$

$$\vdots$$

$$d_z = -s_4 s_5 d_6 + d_1 + d_2$$

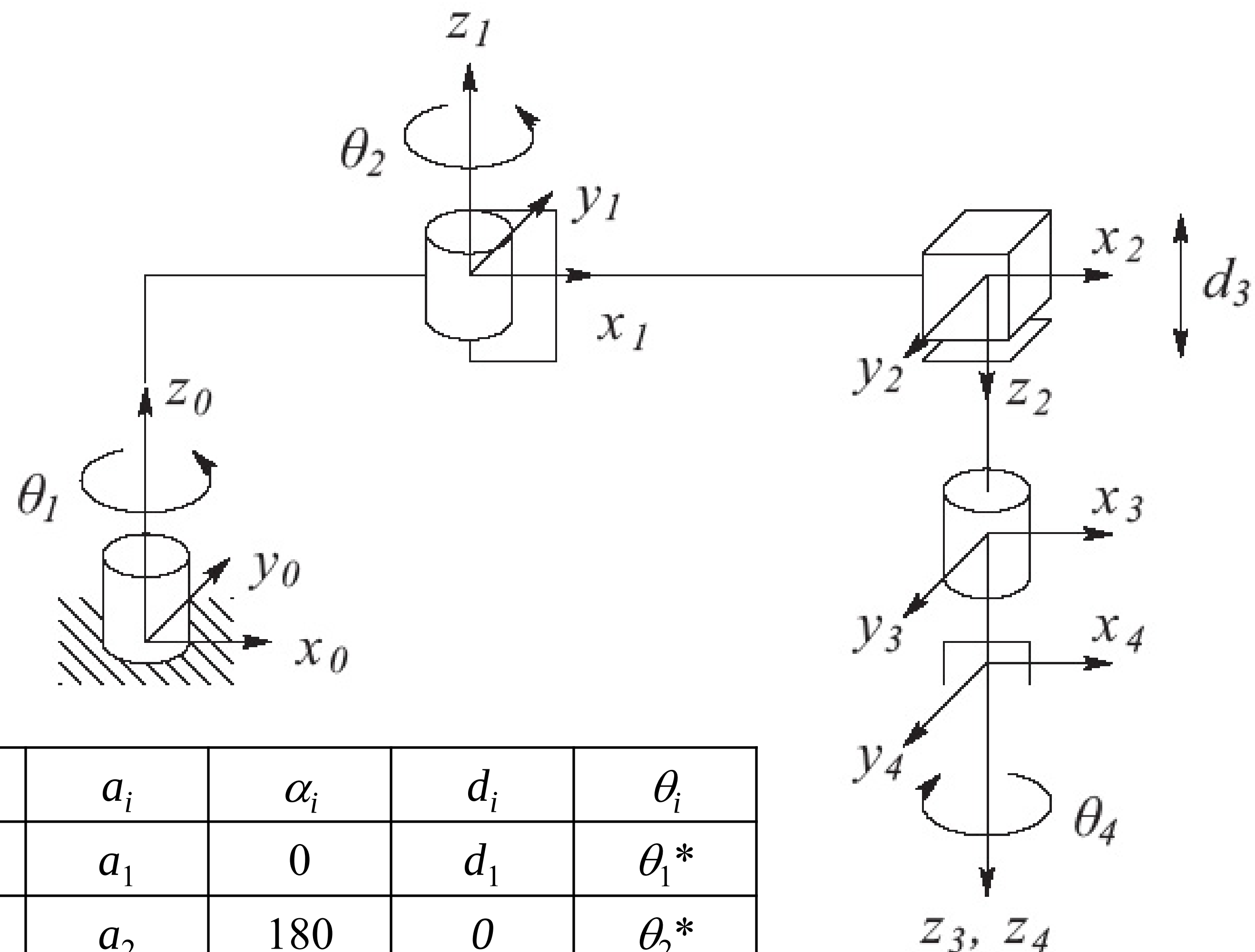
Stanford Manipulator + Spherical Wrist



Link	a_i	α_i	d_i	θ_i
1	0	-90	0	θ_1*
2	0	90	d_2	θ_2*
3	0	0	d_3*	0
4	0	-90	*0*	θ_4*
5	0	90	*0*	θ_5*
6	0	0	d_6	θ_6*

* joint variable

SCARA + 1DOF Wrist



Link	a_i	α_i	d_i	θ_i
1	a_1	0	d_1	θ_1*
2	a_2	180	0	θ_2*
3	0	0	d_3*	0
4	0	0	d_4	θ_4*

* joint variable